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Random Mann iteration scheme and random fixed point theorems

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Abstract

In this paper, some concepts such as random monotone operators, random Mann iteration and so on in a separable real Banach space are introduced. Also the existence and uniqueness theorems of random fixed points for random monotone operators satisfying *Condition*(H) are proved.

Keywords: Random Mann iteration; Random increasing (decreasing) operators; Random fixed point; Normal cone; Banach space

1. Introduction

Probabilistic functional analysis has come out as one of the momentous mathematical disciplines in view of its requirements in dealing with probabilistic models in applied problems. The study of random fixed points forms a central topic in this area. Random fixed point theory has received much attention since the publication of the survey article by Bharuch-Reid [3] in 1976, in which the stochastic version of some well-known fixed point theorems were proved. Since then there has been a lot of activity in this area. For example, Li [5,6,8] has introduced the random fixed point index theory and obtained some excellent random fixed point theorems which are applied to investigate the existence of random solutions for random Hammerstein equation (see [7,9] for details). Furthermore, Itoh [10], O'Regan [11]

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and Shahzad [12,13] studied many random fixed points of contractive random maps satisfying different conditions. Recently, Yu and Guo [14] studied the Mann iteration scheme (also see Mann [1]). Choudhury [4] constructed a random Mann iteration scheme in a separable Hilbert space and proved a random fixed point theorem satisfying a certain contractive inequality. In conclusion, random fixed point theorems in connection with random approximations are studied extensively. In this paper, we first introduce some new concepts such as random monotone operators, random Mann iteration and so on, which are a little different from those in [4]. Moreover, we consider random operators satisfying *Condition*(H) and prove some new random fixed theorems on these operators by virtue of monotone iterative methods and partial ordered theory. In fact, we also obtain the uniqueness of random fixed points, which almost can not be obtained by the methods in previous literatures.

2. Preliminaries

Throughout this paper, let (Ω, Σ, μ) be a complete measure space, E a separable real Banach space, (E, β) a measurable space, where β denotes the σ-algebra of all Borel subsets generated by all open subsets in E. C is a nonempty subset of E.

A mapping $T : \Omega \to C$ is called measurable if $T^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of E.

A mapping $T : \Omega \times C \to C$ is said to be a random operator if for each fixed $x \in C$, the map $T(\cdot, x) : \Omega \to C$ is measurable.

A measurable map $\xi : \Omega \to C$ is called a random fixed point of the random operator $T : \Omega \times C \to C$, if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$.

A random operator $T : \Omega \times C \to C$ is said to be continuous if for any fixed $\omega \in \Omega$, $T(\omega, \cdot) : C \to C$ is continuous.

Definition 1. A cone P in E is said to be normal if there exists a constant $N > 0$ such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$, i.e., the norm $\|\cdot\|$ is semi-monotone, where N is called the normal constant of P (see [15,16] for details).

Definition 2. A mapping $T : \Omega \times C \to C$, where C is a nonempty subset of a real Banach space, is called a random increasing operator for any fixed $\omega \in \Omega$, $T(\omega, \cdot)$ is increasing and for any fixed $x \in C$, $T(\cdot, x)$ is measurable.

Similarly, we can define a random decreasing operator.

Definition 3 (*Random Mann Iteration Scheme*). Let $T : \Omega \times C \to C$ be a random operator, where C is a nonempty convex subset of a real separable Banach space E. Then the sequence of mappings $\{g_n\}$ is defined as in the following:

$$\text{Let} \quad g_0 : \Omega \to C \text{ be an arbitrary measurable mapping.} \tag{1}$$

For $\omega \in \Omega, n = 0, 1, \ldots,$

$$g_{n+1}(\omega) = c_n g_n(\omega) + (1 - c_n)T(\omega, g_n(\omega)), \tag{2}$$

$$\text{where} \quad 0 \le c_n < 1, c_n \le c_{n+1}, \quad n = 0, 1, 2, \ldots, \tag{3}$$

$$\text{and} \quad 0 \le \lim_{n\to\infty} c_n = c < 1. \tag{4}$$

Since C is convex, it follows from the above construction that g_n is a mapping from Ω to C for all $n = 0, 1, 2, \ldots$. In particular, if $c_n = 0$ $(n = 0, 1, 2, \ldots)$, the sequence, i.e., $g_{n+1}(\omega) = T(\omega, g_n(\omega))$, is said to be a *Random Picard Iteration Scheme*.

Definition 4 (*Condition(H)*). A mapping $T : C \to C$, where C is a nonempty subset of a real Banach space, is said to satisfy *Condition*(H) if for all $x, y \in C, y \le x$, there exists a positive linear bounded operator $L : E \to E$ such that

$$-L(x-y) \le Tx - Ty \le L(x-y) \quad \text{and} \quad \|L\| < 1. \tag{5}$$

Remark. When T is an increasing operator, (5) is equivalent to

$$Tx - Ty \le L(x-y) \quad \text{and} \quad \|L\| < 1. \tag{6}$$

When T is a decreasing operator, (5) is equivalent to

$$Ty - Tx \le L(x-y) \quad \text{and} \quad \|L\| < 1. \tag{7}$$

3. Main results

For any fixed u_0 in E, let $g_0 \equiv u_0$, $C = \{x \in E | x \ge g_0\}$. It is clear that $g_0 : \Omega \to C$ is a measurable mapping and C is a nonempty convex subset of E.

Theorem 3.1. *Let E be a separable real Banach space, P a normal cone in E. Assume that $T : \Omega \times C \to C$ is a random operator which satisfies the following conditions:*

(1) *T is a continuous random increasing operator,*
(2) *for any $\omega \in \Omega$, $T(\omega, \cdot) : C \to C$ satisfies Condition(H).*

Then the random Mann iteration scheme constructed in (1)–(4) converges to a unique random fixed point ξ of T. Moreover, for any initial $x_0 \in C$, $\omega \in \Omega$, set $x_{n+1}(\omega) = c_n x_n(\omega) + (1 - c_n)T(\omega, x_n(\omega))$, $n = 0, 1, 2, \ldots$, $x_n(\omega) \to \xi(\omega)$, as $n \to \infty$. In addition, for any h satisfying $\|L\| \le h < 1$, we have

$$\|x_n(\omega) - \xi(\omega)\| \le N[h + c(1-h)]^n \left[\|x_0 - g_0\| + \frac{1}{(1-c)(1-h)} \|g_1(\omega) - g_0\| \right], \quad n \ge 0. \tag{8}$$

Proof. For any fixed $\omega \in \Omega$, set

$$g_{n+1}(\omega) = c_n g_n(\omega) + (1 - c_n)T(\omega, g_n(\omega)), n = 0, 1, \ldots. \tag{9}$$

Next we divide the proof of Theorem 3.1 into five steps.

(a) By induction, we have

$$g_n(\omega) \in C, \qquad T(\omega, g_n(\omega)) \in C, \tag{10}$$

$$g_n(\omega) \le T(\omega, g_n(\omega)), \quad n = 0, 1, 2, \ldots, \tag{11}$$

and it is easy to prove that

$$\{g_n(\omega)\} \quad \text{and} \quad \{T(\omega, g_n(\omega))\} \quad \text{are monotone increasing sequences.} \tag{12}$$

(b) There exists $u^*(\omega)$, $v^*(\omega)$ in C such that $g_n(\omega) \to u^*(\omega)$, $T(\omega, g_n(\omega)) \to v^*(\omega)$, as $n \to \infty$. In fact, for any natural number $n \ge 1$, by Remark (6), we have

$$\begin{aligned}
&(g_{n+1}(\omega) - g_n(\omega)) - (g_n(\omega) - g_{n-1}(\omega)) \\
&\quad = (1 - c_n)(T(\omega, g_n(\omega)) - g_n(\omega)) - (1 - c_{n-1})(T(\omega, g_{n-1}(\omega)) - g_{n-1}(\omega)) \\
&\quad \le (1 - c_{n-1})(L - I)(g_n(\omega) - g_{n-1}(\omega)).
\end{aligned}$$

And so

$$\begin{aligned} g_{n+1}(\omega) - g_n(\omega) &\le [I + (1 - c_{n-1})(L - I)](g_n(\omega) - g_{n-1}(\omega)) \\ &= [L(1 - c_{n-1}) + c_{n-1} I](g_n(\omega) - g_{n-1}(\omega)). \end{aligned} \tag{13}$$

By using (13) repeatedly, it follows that

$$\begin{aligned} \theta &\le g_{n+1}(\omega) - g_n(\omega) \\ &\le [L(1 - c_{n-1}) + c_{n-1} I](g_n(\omega) - g_{n-1}(\omega)) \\ &\le \cdots \le K_{n-1} \cdot K_{n-2} \cdots K_0 \cdot (g_1(\omega) - g_0), \end{aligned} \tag{14}$$

where $K_i = L(1 - c_i) + c_i I$, $i = 0, 1, 2, \ldots$. For any h satisfying $\|L\| \le h < 1$, we have

$$\begin{aligned} \|K_i\| = \|L(1 - c_i) + c_i I\| &\le h(1 - c_i) + c_i \\ &= h + c_i(1 - h) \le h + c(1 - h) < 1. \end{aligned} \tag{15}$$

It follows from the normality of P, (14) and (15), that

$$\|g_{n+1}(\omega) - g_n(\omega)\| \le N[h + c(1 - h)]^n \|g_1(\omega) - g_0\|. \tag{16}$$

And so, we obtain, for any natural number p,

$$\begin{aligned} \|g_{n+p}(\omega) - g_n(\omega)\| &\le \sum_{i=1}^{p} \|g_{n+i}(\omega) - g_{n+i-1}(\omega)\| \\ &\le \sum_{i=1}^{p} N\alpha^{n+i-1} \|g_1(\omega) - g_0\| \\ &\le N\|g_1(\omega) - g_0\| \alpha^n \frac{(1 - \alpha)^p}{(1 - \alpha)} \\ &\le N\|g_1(\omega) - g_0\| \frac{\alpha^n}{(1 - \alpha)}, \end{aligned} \tag{17}$$

where $\alpha = h + c(1 - h)$. Therefore, $\{g_n(\omega)\}$ is a Cauchy sequence in E. Notice that E is a Banach space and $g_n(\omega)$ is in C, so there exists $u^*(\omega)$ in C such that $g_n(\omega) \to u^*(\omega)$, as $n \to \infty$. Taking $p \to \infty$ in (17), we have

$$\|u^*(\omega) - g_n(\omega)\| \le N\|g_1(\omega) - g_0\| \frac{\alpha^n}{(1 - \alpha)}. \tag{18}$$

Note that T is a random increasing operator, by Remark (6), we have $\theta \le T(\omega, g_{n+1}(\omega)) - T(\omega, g_n(\omega)) \le L(g_{n+1}(\omega) - g_n(\omega))$. It thus follows from the normality of P that $\|T(\omega, g_{n+1}(\omega)) - T(\omega, g_n(\omega))\| \le N\|L\|\|g_{n+1}(\omega) - g_n(\omega)\| \le Nh\|g_{n+1}(\omega) - g_n(\omega)\|$. As with the proof of (16) and (17), it follows that $\{T(\omega, g_n(\omega))\}$ is also a Cauchy sequence, which implies that there exists $v^*(\omega)$ in C such that $T(\omega, g_n(\omega)) \to v^*(\omega)$, as $n \to \infty$.

(c) The existence of a random fixed point. Taking $n \to \infty$ in (9), we obtain $u^*(\omega) = cu^*(\omega) + (1 - c)v^*(\omega)$, which implies $u^*(\omega) = v^*(\omega)$. By virtue of the continuity of $T(\omega, \cdot)$, we know that $T(\omega, g_n(\omega))$ converges to $T(\omega, u^*(\omega))$, and so, $T(\omega, u^*(\omega)) = v^*(\omega) = u^*(\omega) \stackrel{\triangle}{=} \xi(\omega)$, i.e., $T(\omega, \xi(\omega)) = \xi(\omega)$ and $\xi(\omega)$ in C. In addition, since T is a continuous random operator, for any measurable mapping f from Ω to C, $T(\omega, f(\omega))$ is also a measurable mapping [2]. It thus follows from (1)–(4) that $\{g_n\}$ is a sequence of measurable mappings. Hence, $\xi : \Omega \to C$, being the limit of a sequence of measurable mappings, is also measurable. So $\xi : \Omega \to C$ is a random fixed point of the random operator T.

(d) We will prove the uniqueness of random fixed points. Suppose that there exists another $\zeta : \Omega \to C$ such that $T(\omega, \zeta(\omega)) = \zeta(\omega)$. From $\zeta(\omega) \geq g_0$, by induction, it is not difficult to obtain that $\zeta(\omega) \geq g_n(\omega)$, $n = 0, 1, \ldots$, and so

$$\theta \leq \zeta(\omega) - g_n(\omega) \leq K_{n-1} \cdots K_0(\zeta(\omega) - g_0).$$

As a result,

$$\|\zeta(\omega) - g_n(\omega)\| \leq N\|K_{n-1}\| \cdots \|K_0\|\|(\zeta(\omega) - g_0)\|. \tag{19}$$

Therefore, $g_n(\omega) \to \zeta(\omega)$, as $n \to \infty$, which implies $\xi(\omega) = \zeta(\omega)$.

(e) For any x_0 in C, ω in Ω, set $x_{n+1}(\omega) = c_n x_n(\omega) + (1 - c_n)T(\omega, x_n(\omega))$, we have $x_{n+1}(\omega) \to \xi(\omega)$, as $n \to \infty$. In fact, it follows from x_0 in C that $x_0 \geq g_0$. By induction, it is easy to prove that $x_n(\omega) \geq g_n(\omega)$, $n = 0, 1, \ldots$, and so

$$\theta \leq x_n(\omega) - g_n(\omega) \leq K_{n-1}K_{n-2} \cdots K_0(x_0 - g_0).$$

Since P is a normal cone, we obtain

$$\|x_n(\omega) - g_n(\omega)\| \leq N\|K_{n-1}\| \cdots \|K_0\|\|x_0 - g_0\| \leq N\alpha^n\|x_0 - g_0\|. \tag{20}$$

It follows from (18) and (20) that

$$\begin{aligned}\|x_n(\omega) - \xi(\omega)\| &\leq \|x_n(\omega) - g_n(\omega)\| + \|g_n(\omega) - \xi(\omega)\| \\ &\leq N\alpha^n[\|x_0 - g_0\| + \tfrac{1}{1-\alpha}\|g_1(\omega) - g_0\|].\end{aligned} \tag{21}$$

Taking $n \to \infty$ in (21), it follows that $x_n(\omega) \to \xi(\omega)$, and (8) holds. This completes the proof of Theorem 3.1. □

Corollary 3.1. *Let E, P, C be as in Theorem 3.1, Assume that $T : \Omega \times C \to C$ is a random operator which satisfies the following conditions:*

(1) *T is a continuous random increasing operator,*
(2) *for any $\omega \in \Omega$, $T(\omega, \cdot) : C \to C$ satisfies Condition(H).*

Then for an arbitrary x_0 in C, the iteration scheme defined by

$$x_{n+1}(\omega) = T(\omega, x_n(\omega)), \quad n = 0, 1, 2, \ldots, \omega \in \Omega$$

converges to the unique random fixed point $\xi : \Omega \to C$ of T.

For any fixed v_0 in E, let $g_0 \equiv v_0$, $C = \{x \in E | x \leq g_0\}$. It is clear that $g_0 : \Omega \to C$ is a measurable mapping and C is a nonempty convex subset of E. Similarly, we can obtain the following theorem by the same method:

Theorem 3.2. *Let E be a separable real Banach space, P a normal cone in E. Assume that $T : \Omega \times C \to C$ is a random operator which satisfies the following conditions:*

(1) *T is a continuous random increasing operator,*
(2) *for any $\omega \in \Omega$, $T(\omega, \cdot) : C \to C$ satisfies Condition(H).*

Then the random Mann iteration scheme constructed in (1)–(4) *converges to a unique random fixed point* ξ *of* T. *Moreover, for any* $x_0 \in C$, $\omega \in \Omega$, *set* $x_{n+1}(\omega) = c_n x_n(\omega) + (1 - c_n)T(\omega, x_n(\omega))$, $n = 0, 1, 2, \ldots$, $x_n(\omega) \to \xi(\omega)$, *as* $n \to \infty$. *In addition, for any* h *satisfying* $\|L\| \le h < 1$, *we have*

$$\|x_n(\omega) - \xi(\omega)\| \le N[h + c(1-h)]^n \left[\|x_0 - g_0\| + \frac{1}{(1-c)(1-h)}\|g_1(\omega) - g_0\|\right], n \ge 0. \tag{22}$$

Corollary 3.2. *Let* E, P, C *be the same as in Theorem 3.2, Suppose that* $T : \Omega \times C \to C$ *is a random operator which satisfies the following conditions:*

(1) T *is a continuous random increasing operator,*
(2) *for any* $\omega \in \Omega$, $T(\omega, \cdot) : C \to C$ *satisfies Condition(H).*

Then for an arbitrary x_0 *in* C, *the iteration scheme defined by*

$$x_{n+1}(\omega) = T(\omega, x_n(\omega)), \quad n = 0, 1, 2, \ldots, \omega \in \Omega$$

converges to the unique random fixed point $\xi : \Omega \to C$ *of* T.

Theorem 3.3. *Let* P *be a normal cone of a separable real Banach space* E. *Assume that* $T : \Omega \times [u_0, v_0] \to [u_0, v_0]$ *is a random operator which satisfies the following conditions:*

(1) T *is a continuous random decreasing operator,*
(2) *for any* $\omega \in \Omega$, $T(\omega, \cdot) : [u_0, v_0] \to [u_0, v_0]$ *satisfies Condition(H).*

Then the random operator T *has exactly one random fixed point* $u^* : \Omega \to [u_0, v_0]$. *Moreover, for any* x_0 *in* $[u_0, v_0]$, *set* $x_{n+1}(\omega) = T(\omega, x_n(\omega))(n = 1, 2, \ldots)$, *we have* $\lim_{n\to\infty} x_{n+1}(\omega) = u^*(\omega)$.

Proof. For any fixed $\omega \in \Omega$, we set

$$u_n(\omega) = T(\omega, v_{n-1}(\omega)), \qquad v_n(\omega) = T(\omega, u_{n-1}(\omega)), \quad n = 1, 2, \ldots. \tag{23}$$

Since $T : \Omega \times [u_0, v_0] \to [u_0, v_0]$, by induction, for any fixed $\omega \in \Omega$ we have

$$u_{n-1}(\omega) \le u_n(\omega) \le v_n(\omega) \le v_{n-1}(\omega), \quad n = 1, 2, \ldots. \tag{24}$$

From Remark (7), for $n \ge 2$, we obtain

$$\begin{aligned}\theta &\le u_{n+1}(\omega) - u_n(\omega) = T(\omega, v_n(\omega)) - T(\omega, v_{n-1}(\omega)) \\ &\le L(v_{n-1}(\omega) - v_n(\omega)) \le L^2(u_{n-1}(\omega) - u_{n-2}(\omega)) \\ &\le \cdots \le \begin{cases} L^{2k}(u_1(\omega) - u_0) & \text{when} \quad n = 2k, \\ L^{2k}(u_2(\omega) - u_1(\omega)) & \text{when} \quad n = 2k+1. \end{cases}\end{aligned} \tag{25}$$

It follows from the normality of P that

$$\|u_{n+1}(\omega) - u_n(\omega)\| \le N\|L\|^{n-1}R, \tag{26}$$

where $R = \max\{\|u_1(\omega) - u_0\|, \|u_2(\omega) - u_1(\omega)\|\}$. And so, for any natural number p, we have

$$\begin{aligned}\|u_{n+p}(\omega) - u_n(\omega)\| &\le \sum_{i=1}^{p} \|u_{n+i}(\omega) - u_{n+i-1}(\omega)\| \\ &\le NR \cdot \sum_{i=1}^{p} \|L\|^{n+i-2} \le NR \cdot \frac{\|L\|^{n-1}}{1 - \|L\|}.\end{aligned} \tag{27}$$

So the sequence $\{u_n(\omega)\}$ is a Cauchy sequence in E, which shows that there exists $x^*(\omega)$ in $[u_0, v_0]$ such that $u_n(\omega)$ converges to $x^*(\omega)$. Similarly, there exists $x_*(\omega)$ in $[u_0, v_0]$ such that $v_n(\omega)$ converges to $x_*(\omega)$. Next we will prove $x^*(\omega) = x_*(\omega)$. In fact, for any n, we have from Remark (7)

$$\begin{aligned}\theta &\le v_n(\omega) - u_n(\omega) = T(\omega, u_{n-1}(\omega)) - T(\omega, v_{n-1}(\omega)) \\ &\le L(v_{n-1}(\omega) - u_{n-1}(\omega)) \le \cdots \le L^n(v_0 - u_0).\end{aligned} \tag{28}$$

And so, $\|v_n(\omega) - u_n(\omega)\| \to 0$, as $n \to \infty$, i.e., $u^*(\omega) \overset{\triangle}{=} x^*(\omega) = x_*(\omega)$.

It thus follows from (23) and condition (1) that $\{u_n\}$ is a sequence of measurable mappings. Hence $u* : \Omega \to [u_0, v_0]$, being the limit of a sequence of measurable mappings, is also measurable. So $u* : \Omega \to [u_0, v_0]$ is a random fixed point of T.

It is not difficult to prove the remainders of Theorem 3.3. Since the method is the same as the proof of Theorem 3.1, we omit the proof. □

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